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Nonlinear critical layers in the boundary layer on a rotating disk

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Abstract The work of Gregory, Stuart and Walker (1955, Proc R Soc Ser A 406:93–106) and Hall (1986, Phil Trans R Soc London Ser A 248:155–199) is extended to include nonlinear effects for the stationary cross-flow vortex. It is shown that amplitude-dependent neutral modes are described by a forced Haberman equation. The corrections to the neutral wavenumbers and waveangles are derived and it is suggested that the nonlinear neutral modes can have wavenumbers decreased by an O(1) amount as compared to linear theory.

Keywords Hydrodynamic stability · Rotating disk flow · Cross-flow · Nonlinear

1 Introduction

In one of the first detailed experimental and theoretical investigations of three-dimensional boundary-layer stability, Gregory, Stuart and Walker [1] (hereafter referred to as GSW), highlighted the importance of the cross-flow mechanism and its relevance to the stationary mode observed in rotating-disk flow. Cross-flow instabilities are characteristic of a fully three-dimensional boundary-layer flow such as that occurring on a swept wing. The basic mechanism involved is that, in appropriate flow directions, the effective velocity profile appears to be inflexional, and hence prone to inflexional instability. One particular profile is such that the point of inflexion coincides with the point of zero velocity, thus giving rise to neutral disturbances with zero phase speed, and commonly termed the stationary mode.

The paper by GSW [1] outlined the basic inviscid theory for the stationary mode. Neutral curves calculated in Malik [2] for the stationary mode showed that, for large Reynolds number, the neutral curve had two distinct branches, one the upper-branch corresponding to the inviscid mode of GSW, and another which was termed the lower-branch mode. Hall [3], using asymptotic methods, obtained the correction terms to the inviscid mode of GSW and showed also how the lower-branch mode of [2] could be identified with wall modes.

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An asymptotic description of the linear and nonlinear properties of non-stationary modes is given in [4]. In [5], the properties of unsteady nonlinear critical layers are used to study the evolution of growing modes.

Our aim in this paper is to extend the work by GSW and that of [3] to include nonlinear effects for the upper-branch stationary mode. Nonlinear calculations for the lower-branch mode are given in [6].

It is clear from the work of [3] (hereafter referred to as I), that the importance of the nonlinear terms will be enhanced in the critical layer. The structure of the solution given in [3] is very similar to that occurring in two-dimensional boundary-layer flows. The extension to include nonlinear effects is therefore very similar to the studies of nonlinear critical layers in [7–12]. In the canonical problem, however, there are differences. The governing equation here turns out to be a forced Haberman equation, see [7], the forcing arising from the Coriolis terms. The other difference from two-dimensional flows is the important role played by the mean-flow interaction terms. As in I, the corrections to the wavenumber and wave-angle stem from a balancing of the wall-layer and critical-layer phase shifts. In the Haberman stage these corrections are amplitude-dependent. With increased nonlinearity, the critical-layer phase shift approaches zero and the structure of the modes is similar to that discussed in [8] for two-dimensional flows. In this limit it is found that there are O(1) corrections to the wavenumber and wave-angle. A significant result here is that there is a cross-over from the inflexional profile to one in which U_B'' is non-zero at the critical layer. Here $U_B(y)$ is the effective cross-flow profile.

In Sect. 2 we discuss briefly the critical-layer details for the structure given in [3]. The nonlinear theory is described in Sect. 3. Throughout this work the Reynolds number R is taken to be large.

2 Basic equations and linear theory

Consider a disk, located at $z = 0$, which rotates about the z-axis with angular velocity Ω. With axes rotating with the disk, the Navier–Stokes and continuity equations, suitably non-dimensionalised, are:

$$\mathbf{u}.\nabla\mathbf{u} + 2(\hat{\mathbf{k}} \times \mathbf{u}) - 2r\,\hat{\mathbf{r}} = -\nabla p + \frac{1}{R}\nabla^2\mathbf{u}, \tag{1}$$

$$\nabla.\mathbf{u} = 0. \tag{2}$$

Here $\mathbf{u} = (u, v, w)$ are the velocity components in cylindrical polar coordinates (r, θ, z) and $(\hat{\mathbf{r}}, \hat{\theta}, \hat{\mathbf{k}})$ are the corresponding unit vectors in the respective coordinate directions. Also $R = \Omega^2 l/\nu$ is the Reynolds number, l is some reference lengthscale, and ν is the kinematic viscosity of the fluid.

The basic steady flow is given by the von Kármán [13] solution as,

$$\mathbf{u} = \mathbf{U_B} = (r\bar{u}(\zeta), r\bar{v}(\zeta), R^{-\frac{1}{2}}\bar{w}(\zeta)),$$

where $z = R^{-\frac{1}{2}}\zeta = \epsilon^3\zeta, \quad \epsilon = R^{-\frac{1}{6}}$ and $\bar{u}, \bar{v}, \bar{w}$ satisfy,

$$\bar{u}^2 - (\bar{v}+1)^2 + \bar{u}'\bar{w} - \bar{u}'' = 0,$$
$$2\bar{u}(\bar{v}+1) + \bar{v}'\bar{w} - \bar{v}'' = 0,$$
$$2\bar{u} + \bar{w}' = 0.$$

Primes above denote differentiation with respect to ζ. The boundary conditions on $\mathbf{U_B}$ are

$$\bar{u} = 0, \quad \bar{v} = 0, \quad \bar{w} = 0, \quad \text{at} \quad \zeta = 0,$$
$$\bar{u} \to 0, \bar{v} \to -1 \quad \text{as} \quad \zeta \to \infty.$$

2.1 Linear theory

Following GSW and I, we perturb the basic flow by writing

$$(\mathbf{u}, p) = (\mathbf{U_B}, p_B) + \delta(U, V, W, P)\ldots, \tag{3}$$

where $\delta \ll 1$ is the infinitesimal disturbance size and p_B denotes the basic pressure. For linear theory to hold, the requirement that $\delta \ll O(\epsilon^2)$ is needed to ensure that nonlinear balances in the critical-layer region remain negligible; see later. The disturbance quantities (U, V, W, P) are now expressed as

$$\begin{aligned}
U &= U_0 \cos x + \epsilon U_1 + \cdots, \\
V &= V_0 \cos x + \epsilon V_1 + \cdots, \\
W &= W_0 \sin x + \epsilon W_1 + \cdots, \\
P &= P_0 \cos x + \epsilon P_1 + \cdots,
\end{aligned} \tag{4}$$

where, as in I, we have introduced x-variations such that

$$\frac{\partial}{\partial r} = \epsilon^{-3}(\alpha_0 + \epsilon\alpha_1 + \cdots)\frac{\partial}{\partial x} + \frac{\partial}{\partial r},$$

$$\frac{\partial}{\partial \theta} = \epsilon^{-3}(m_0 + \epsilon m_1 + \cdots)\frac{\partial}{\partial x}.$$

Here $\alpha = \alpha_0 + \epsilon\alpha_1 + \cdots$, $m/r = (m_0 + \epsilon m_1 + \cdots)/r$ are the scaled wavenumbers in the $\hat{\mathbf{r}}, \hat{\theta}$ directions, respectively. The x-variable here accounts for fast variations in the direction of propagation of the wave which makes an angle $\tan^{-1}(\beta/r\alpha)$ with the radial direction. The disturbance eigenfunctions U_0, V_0, W_0, P_0 all depend on ζ (as well as r), but not on x. In later sections the explicit dependence on r is suppressed. The main objective of our work is to discuss how the neutral wavenumbers and waveangles change with amplitude of the disturbance.

Substitution of (4) in the Navier–Stokes and continuity equations yields at leading order,

$$\begin{aligned}
-U_{0B}U_0 + rW_0\bar{u}' &= \alpha_0 P_0, \\
-U_{0B}V_0 + rW_0\bar{v}' &= \tfrac{m_0}{r} P_0, \\
-U_{0B}W_0 &= -P_0', \\
-U_{00} + W_0' &= 0,
\end{aligned} \tag{5}$$

where primes denote differentiation with respect to ζ, and we have defined $U_{nB} = r\alpha_n\bar{u} + m_n\bar{v}$ and $U_{nk} = \alpha_n U_k + (m_n/r)V_k$. Thus U_{0B} can be regarded as the 'effective' or cross-flow velocity. Manipulation of (5) gives the Rayleigh equation for W_0, namely:

$$U_{0B}[W_0'' - \gamma_0^2 W_0] - U_{0B}'' W_0 = 0, \tag{6a}$$

where $\gamma_0^2 = \alpha_0^2 + (m_0^2/r)$ is the effective wavenumber.

We are interested in stationary modes, so following GSW and I, we choose α_0, m_0 such that

$$U_{0B} = U_{0B}'' = 0 \quad \text{at} \quad \zeta = \bar{\zeta}. \tag{6b}$$

The equation for W_0, (6a) together with (6b) and the boundary conditions

$$W_0 = 0 \quad \text{at} \quad \zeta = 0, \infty, \tag{6c}$$

specifies an eigenvalue problem for determining γ_0. The solution is given in I and it is noted that

$$\gamma_0 = 1.16, \quad \alpha_0/m_0 = 4.26/r, \quad \bar{\zeta} = 1.46.$$

The behaviour of the eigenfunctions locally near $\zeta = \bar{\zeta}$ are required in the subsequent analysis and it is easily deduced that

$$U_0 \sim -\frac{[\alpha_0 P_0 - rW_0\bar{u}']}{B_{01}(\zeta - \bar{\zeta})} + \cdots, \tag{7a}$$

$$V_0 \sim -\frac{[(m_0/r)P_0 - rW_0\bar{v}']}{B_{01}(\zeta - \bar{\zeta})} + \cdots, \tag{7b}$$

as $\zeta \to \bar{\zeta}$. The constants B_{jk} are defined by

$$B_{jk} = r\alpha_j a_{ku} + m_j a_{kv},$$

where

$$a_{ku} = \left(\frac{\mathrm{d}^k\bar{u}}{\mathrm{d}\zeta^k}\right)_{\zeta=\bar{\zeta}}, \quad a_{kv} = \left(\frac{\mathrm{d}^k\bar{v}}{\mathrm{d}\zeta^k}\right)_{\zeta=\bar{\zeta}}.$$

Note that, whereas U_0, V_0 are singular at the critical layer, W_0 and U_{00} are regular at $\zeta = \bar{\zeta}$. From (6b), $B_{00} = B_{02} = 0$. The eigenfunction W_0 is normalised such that $W_0'(\zeta = 0) = 1$.

The viscous terms in the full equations are needed to smooth out the singularities at the critical level $\zeta = \bar{\zeta}$ and a balance of inertial and viscous terms shows that the thickness of the critical layer is $\mathrm{O}(\epsilon)$.

At the next order in the inviscid region, substituting (4) in the Navier–Stokes equations and writing

$$W_1 = W_{1s}(\zeta)\sin x + W_{1c}(\zeta)\cos x,$$

we find that W_{1c} satisfies the same equation as W_0. However, W_{1s} satisfies

$$U_{0B}[W_{1s}'' - \gamma_0^2 W_{1s}] - U_{0B}'' W_{1s} = 2U_{0B}(\alpha_0\alpha_1 + m_0 m_1)W_0 + \left(\alpha_1 - \frac{m_1\alpha_0}{m_0}\right) r\left(\bar{u}'' - \frac{U_{0B}''\bar{u}}{U_{0B}}\right) W_0. \tag{8}$$

The second term on the right-hand side of (8) gives a logarithmic singularity for W_{1s} with the behaviour

$$W_{1s}'' \sim r\frac{(m_0\alpha_1 - m_1\alpha_0)(B_{01}a_{2u} - B_{03}a_{0u})}{m_0 B_{01}^2(\zeta - \bar{\zeta})}(W_0)_{\zeta=\bar{\zeta}}, \tag{9}$$

as $\zeta \to \bar{\zeta}+$. The continuation below the critical layer ($\zeta < \bar{\zeta}$), see Sect. 3.1 below, gives the well-known phase jump

$$W_1 \sim k_0(\zeta - \bar{\zeta})\log(\zeta - \bar{\zeta})\sin x \quad \zeta > \bar{\zeta},$$
$$W_1 \sim k_0(\zeta - \bar{\zeta})[\log(\zeta - \bar{\zeta})\sin x + \pi\cos x] \quad \zeta < \bar{\zeta},$$

where k_0 is the coefficient of $(\zeta - \bar{\zeta})^{-1}$ in (9). Hence

$$[W_{1c}']_-^+ = k_0\pi, \quad [W_{1s}']_-^+ = 0, \tag{10}$$

with $[]_-^+$ denoting the jump across $\zeta = \bar{\zeta}$.

Before considering the critical layer, the wall-layer problem, see I, where $\zeta = \epsilon^4 \xi$ and

$$U = \bar{U}_0 + \cdots, \quad V = \bar{V}_0 + \cdots, \quad W = \epsilon \bar{W}_0 + \cdots, \tag{11}$$

shows that W_0 satisfies a forced Airy equation and, in particular, the solution properties imply that

$$W_0 \sim \frac{\xi\lambda}{r} \sin x - \frac{3\mathrm{Ai}'(0)}{\lambda^{\frac{1}{3}}} \cos(x + \pi/3) \quad \text{as} \quad \xi \gg 1. \tag{12}$$

Here we have defined $\lambda(> 0)$ by $\lambda = \alpha_0 \bar{u}'(0) + (m_0/r)\bar{v}'(0)$. The phase-shift given by (10) matches with the wall-layer phase shift in (12) and gives rise to the eigenrelations. In fact, a solvability condition on W_1 in the inviscid region shows that

$$[W'_{1c} W_0 - W_{1c} W'_0]^{\infty}_{+} + [W'_{1c} W_0 - W_{1c} W'_0]^{-}_{0} = 0, \tag{13a}$$

$$[W'_{1s} W_0 - W_{1s} W'_0]^{\infty}_{+} + [W'_{1s} W_0 - W_{1s} W'_0]^{-}_{0} = 2\left(\alpha_0\alpha_1 + \frac{m_0 m_1}{r^2}\right) \int_0^\infty W_0^2 \, \mathrm{d}\zeta + r m_0 \left(\frac{\alpha_1}{m_0} - \frac{m_1 \alpha_0}{m_0^2}\right) \fint_0^\infty W_0^2 \left(\frac{\bar{u}'' U_{0B} - U''_{0B} \bar{u}}{U_{0B}^2}\right) \mathrm{d}\zeta. \tag{13b}$$

Using (10), (12) and evaluating (13a), we obtain

$$(W_0^2)(\bar{\zeta}) r \left(\alpha_1 - \frac{m_1 \alpha_0}{m_0}\right) \left(a_{2u} - \frac{B_{03} a_{0u}}{B_{01}}\right) \frac{\pi}{B_{01}} = \frac{3\mathrm{Ai}'(0)}{2\lambda^{\frac{1}{3}}}. \tag{14}$$

Similarly, (13b) gives

$$\frac{3\sqrt{3}\mathrm{Ai}'(0)}{2\lambda^{\frac{1}{3}}} = 2\left(\alpha_0\alpha_1 + \frac{m_0 m_1}{r^2}\right) \int_0^\infty W_0^2 \, \mathrm{d}\zeta + \left(\frac{\alpha_1}{m_0} - \frac{\alpha_0 m_1}{m_0^2}\right) r m_0 \fint_0^\infty W_0^2 \left(\frac{\bar{u}'' U_{0B} - U''_{0B} \bar{u}}{U_{0B}^2}\right) \mathrm{d}\zeta. \tag{15}$$

The relations (14, 15) are equivalent to the relations derived by Hall [3]. We note that the jump across the critical layer fixes the correction to the wave-angle, and the conditions on W_{1s} determine the correction to the wave-number.

Our calculations show that

$$I_1 = \int_0^\infty W_0^2 \, \mathrm{d}\zeta = 0.091, \quad I_2 = \fint_0^\infty W_0^2 \left(\frac{\bar{u}'' U_{0B} - U''_{0B} \bar{u}}{U_{0B}^2}\right) \mathrm{d}\zeta = 0.0596,$$

which leads to

$$\alpha_0\alpha_1 + \frac{m_0 m_1}{r^2} = -9.2 r^{-\frac{1}{3}} \gamma_0, \tag{16a}$$

$$\left(\frac{\alpha_1}{m_0} - \frac{m_1 \alpha_0}{m_0^2}\right) r = 17.5 r^{-\frac{1}{3}}. \tag{16b}$$

The numbers in (16a, 16b) are different from I, because in I the jump across the critical layer has the wrong sign in view of the basic flow properties near the critical level.

We turn next to the details of the critical layer.

3 Critical-layer analysis

3.1 Linear critical layer

Let $z = \epsilon^3\bar{\zeta} + \epsilon^4\eta$. For the linear critical layer analysis only, it is more convenient to work in terms of complex quantities and from (7, 9) the expansions in the critical layer region take the form

$$U = (\epsilon^{-1}\tilde{U}_0 + \tilde{U}_1 + \epsilon\tilde{U}_2 + \cdots)\mathrm{e}^{\mathrm{i}x} + \text{c.c.},$$
$$V = (\epsilon^{-1}\tilde{V}_0 + \tilde{V}_1 + \epsilon\tilde{V}_2 + \cdots)\mathrm{e}^{\mathrm{i}x} + \text{c.c.},$$
$$W = (\tilde{W}_0 + \epsilon\tilde{W}_1 + \epsilon^2\tilde{W}_2 + \cdots)\mathrm{e}^{\mathrm{i}x} + \text{c.c.},$$
$$W = (\tilde{P}_0 + \epsilon\tilde{P}_1 + \epsilon^2\tilde{P}_2 + \cdots)\mathrm{e}^{\mathrm{i}x} + \text{c.c.},$$

where c.c. denotes the complex conjugate.

Substitution of the above in (1, 2) yields

$$\tilde{P}_0 = \tilde{P}_1 = \text{const}, \quad \tilde{W}_0 = \frac{-\mathrm{i}\gamma_0^2\tilde{P}_0}{B_{01}}, \quad r\alpha_0\tilde{U}_0 + m_0\tilde{V}_0 = 0,$$

and $\tilde{U}_0$ satisfies the equation

$$\mathrm{i}(B_{01}\eta + B_{10})\tilde{U}_0 - \frac{\mathrm{d}^2\tilde{U}_0}{\mathrm{d}\eta^2} = -\mathrm{i}(\alpha_0\tilde{P}_0 + \tilde{W}_0 r a_{1u}).$$

The solutions for $\tilde{U}_0$ then give the required behaviours (7) as $\eta \to \pm\infty$. At the next order $\tilde{W}_1$ has a trivial solution. $\tilde{W}_2$ satisfies the equation

$$-\mathrm{i}(B_{01}\eta + B_{10})\frac{\mathrm{d}^2\tilde{W}_2}{\mathrm{d}\eta^2} + \frac{\mathrm{d}^4\tilde{W}_2}{\mathrm{d}\eta^4} = -\mathrm{i}\gamma_0^2(B_{01}\eta + B_{10})\tilde{W}_0 + 2\mathrm{i}\gamma_0^2\frac{(1+a_{0v})}{\alpha_0}\frac{\mathrm{d}\tilde{V}_0}{\mathrm{d}\eta} - \mathrm{i}\tilde{W}_0(B_{03}\eta + B_{12}). \tag{17}$$

The second term on the right-hand side of (17) arises from the influence of the Coriolis force and can be removed by writing

$$\frac{\mathrm{d}^2\tilde{W}_2}{\mathrm{d}\eta^2} = \Omega_2 + \left(\gamma_0^2 + \frac{B_{03}}{B_{01}}\right)\tilde{W}_0 + \gamma_0^2\frac{(1+a_{0v})}{B_{01}\alpha_0}\frac{\mathrm{d}^2\tilde{V}_0}{\mathrm{d}\eta^2}.$$

Here Ω_2 satisfies

$$\mathrm{i}(B_{01}\eta + B_{10})\Omega_2 - \frac{\mathrm{d}^2\Omega_2}{\mathrm{d}\eta^2} = \mathrm{i}\tilde{W}_0\left(B_{12} - \frac{B_{03}B_{10}}{B_{01}}\right). \tag{18}$$

The solution for Ω_2,

$$\Omega_2 = \mathrm{i}|B_{01}|^{-2/3}\left(B_{12} - \frac{B_{03}B_{10}}{B_{01}}\right)\int_0^\infty \mathrm{e}^{-t^3/3}\mathrm{e}^{\mathrm{i}(\eta + \frac{B_{10}}{B_{01}})|B_{01}|^{2/3}t}\,\mathrm{d}t$$

gives the continuation below the critical layer and the required '$-\mathrm{i}\pi$' phase jump. This completes the details of the linear critical layer.

We consider next what happens when the disturbance size δ is increased. As in previous critical-layers studies, nonlinear balances will first significantly alter the properties of the critical layer, and this happens when $\delta = \mathrm{O}(\epsilon^2)$.

4 Nonlinear critical layer

4.1 Inviscid region

Here, with $z = \epsilon^3\zeta$, the above discussion suggests that in this region the expansions for the flow quantities will take the form

$$u = r\bar{u} + \epsilon\bar{u}_M(\zeta) + \epsilon^2(\bar{u}_0 + u_{M1}(\zeta)) + \epsilon^3\bar{u}_1 + \epsilon^4\bar{u}_2 + \cdots, \quad (19a)$$
$$v = r\bar{v} + \epsilon\bar{v}_M(\zeta) + \epsilon^2(\bar{v}_0 + v_{M1}(\zeta)) + \epsilon^3\bar{v}_1 + \epsilon^4\bar{v}_2 + \cdots, \quad (19b)$$
$$w = \epsilon^2\bar{w}_0 + \epsilon^3\bar{w}_1 + \cdots, \quad (19c)$$
$$p = \epsilon^2\bar{p}_0 + \epsilon^3\bar{p}_1 + \cdots. \quad (19d)$$

The additional terms $\bar{u}_M, \bar{v}_M$ at $\mathrm{O}(\epsilon)$ are necessary because of the properties of the nonlinear critical layer, see [7, 9, 10]. It is noted also that the disturbance for the w-component of velocity is larger than the basic flow in this direction. Next writing,

$$(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{p}_0) = A(u_0(\zeta)\cos x, v_0(\zeta)\cos x, w_0(\zeta)\sin x, p_0(\zeta)\cos x), \quad (20)$$

and substituting in the Navier–Stokes equations then shows that $w_0(\zeta)$ satisfies the Rayleigh equation (6a) with boundary conditions (6b, 6c). In (20) the amplitude constant A is O(1), given that the disturbances represented by the terms of $(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{p}_0)$ are of $\mathrm{O}(\epsilon^2)$.

At next order the $\cos x$ component of $\bar{w}_1$, say w_{1c}, satisfies the same equation as w_0. The $\sin x$ component of $\bar{w}_0$, w_{1s} now satisfies

$$U_{0B}(w''_{1s} - \gamma_0^2 w_{1s}) - U''_{0B}w_{1s} = 2A\left(\alpha_0\alpha_1 + \frac{m_0m_1}{r^2}\right)W_0 - A\frac{(U_{0M}+U_{1B})}{U_{0B}}\left[\frac{U''_{0B}}{U_{0B}} - \frac{(U''_{0M}+U''_{1B})}{(U_{0m}+U_{1B})}\right]W_0, \quad (21)$$

where $U_{kM} = \alpha_k\bar{u}_M + \frac{m_k}{r}\bar{v}_M$. The appearance of the mean-flow terms in (21) will thus alter the eigenrelations determining the corrections to the wavenumber and wave angle. The eigenrelations will also be affected by the properties of the nonlinear critical layer, and it to this that we turn our attention next.

4.2 Critical-layer expansions

The expansions in the critical layer region are now, with $z = \epsilon^3\bar{\zeta} + \epsilon^4\eta$,

$$u = ra_{0u} + \epsilon\tilde{u}_0 + \epsilon^2\tilde{u}_1 + \epsilon^3\tilde{u}_2 + \cdots, \quad (22a)$$
$$v = ra_{0v} + \epsilon\tilde{v}_0 + \epsilon^2\tilde{v}_1 + \epsilon^3\tilde{v}_2 + \cdots, \quad (22b)$$
$$w = \epsilon^2\tilde{w}_0 + (\epsilon^3\tilde{w}_1 + a_{0w}) + \epsilon^4(\tilde{w}_2 + a_{1w}\eta) + \cdots, \quad (22c)$$
$$p = \epsilon^2\tilde{p}_0 + \epsilon^3\tilde{p}_1 + \epsilon^4\tilde{p}_2 + \cdots. \quad (22d)$$

The $a_{0w}, a_{1w}\eta$ terms in (22c) come from expanding the basic flow for the w-component of velocity. The leading-order problem with $\tilde{u}_{jk} = \alpha_j \tilde{u}_k + \frac{m_j}{r}\tilde{v}_k$ is

$$\frac{\partial \tilde{u}_{00}}{\partial x} + \frac{\partial \tilde{w}_0}{\partial \eta} = 0, \tag{23a}$$

$$(B_{10} + \tilde{u}_{00})\frac{\partial \tilde{u}_0}{\partial x} + \tilde{w}_0 \frac{\partial \tilde{u}_0}{\partial \eta} = -\alpha_0 \frac{\partial \tilde{p}_0}{\partial x} + \frac{\partial^2 \tilde{u}_0}{\partial \eta^2}, \tag{23b}$$

$$(B_{10} + \tilde{u}_{00})\frac{\partial \tilde{v}_0}{\partial x} + \tilde{w}_0 \frac{\partial \tilde{v}_0}{\partial \eta} = -\frac{m_0}{r}\frac{\partial \tilde{p}_0}{\partial x} + \frac{\partial^2 \tilde{v}_0}{\partial \eta^2}, \tag{23c}$$

$$\frac{\partial \tilde{p}_0}{\partial \eta} = 0. \tag{23d}$$

These equations may be solved for the cross-flow component, by taking α_0 times (23b) added to m_0/r times (23c). After matching with the inviscid regions, this gives

$$\tilde{w}_0 = -\frac{\gamma_0^2}{B_{01}}\frac{\partial \tilde{p}_0}{\partial x} = \frac{\gamma_0^2 A p_{00}}{B_{01}} \sin x, \quad \tilde{p}_0 = A p_{00} \cos x, \quad p_{00} = p_0(\bar{\zeta}), \quad \tilde{u}_{00} = B_{01}\eta + C_{00}, \tag{24}$$

and $C_{00} = \alpha_0 u_M^{\pm}(\bar{\zeta}) + (m_0/r) v_M^{\pm}(\bar{\zeta})$ is independent of η and therefore does not jump across $\zeta = \bar{\zeta}$. Writing $\bar{B}_{10} = B_{10} + C_{00}$, $\tilde{v}_0$ satisfies,

$$(\bar{B}_{10} + B_{01}\eta)\frac{\partial \tilde{v}_0}{\partial x} + \tilde{w}_0 \frac{\partial \tilde{v}_0}{\partial \eta} = -\frac{m_0}{r}\frac{\partial \tilde{p}_0}{\partial x} + \frac{\partial^2 \tilde{v}_0}{\partial \eta^2}. \tag{25}$$

Equation 25 can be written in a more standard form with the normalisations:

$$\tilde{v}_0 = r a_{1v}\eta + K_0(Y - V_0^*), \quad K_0 = \left[\frac{m_0}{r} p_{00} A - \frac{\gamma_0^2 A p_{00} r a_{1v}}{B_{01}}\right]\frac{1}{|B_{01}|C_*}, \quad -C_* Y = \frac{\bar{B}_{10}}{B_{01}} + \eta,$$

$$C_*^2 = \frac{\gamma_0^2 A p_{00}}{B_{01}^2}, \quad B_{01}\lambda_c = C_*^{-3}. \tag{26}$$

and then V_0^* satisfies the Haberman [7] nonlinear critical layer equation,

$$Y\frac{\partial V_0^*}{\partial x} + \sin x \frac{\partial V_0^*}{\partial Y} - \lambda_c \frac{\partial^2 V_0^*}{\partial \eta^2} = 0, \tag{27a}$$

with boundary conditions,

$$V_0^* \sim Y + \frac{v_M{}^{\pm}(\bar{\zeta})}{K_0} + \frac{\cos x}{Y} + \cdots \quad \text{as} \quad Y \to \pm\infty, \tag{27b}$$

to match with the inviscid zone. The parameter λ_c arising from the normalisations gives a measure of nonlinearity in the sense that linear theory corresponds to $\lambda_c \to \infty$ and strong nonlinearity when $\lambda_c \to 0$. The properties of Eq. 27 are well known; see [7, 9, 11]. In particular, V_0^* plays the role of the vorticity here, so that with $V_0^* = \partial^2 \psi_0^*/\partial Y^2$, and

$$\frac{\partial \psi_0^*}{\partial Y} \sim \frac{Y^2}{2} + \frac{v_M^{\pm}(\bar{\zeta})}{K_0} Y + \cos x \ln|Y| + B_0^{\pm}(x) \quad \text{as} \quad Y \to \pm\infty.$$

It can be shown that

$$2\lambda_c \frac{\bar{v}_M^+(\bar{\zeta}) - \bar{v}_M^-(\bar{\zeta})}{K_0} = \frac{1}{\pi}\int_0^{2\pi} (B_0^+ - B_0^-)\sin x \, \mathrm{d}x = \mu. \tag{28}$$

In the limit that $A \to 0$, $\lambda_c \to \infty$ and $\mu \to -\pi$ corresponding to linear theory. For $A \gg 1$, $\lambda_c \to 0$ and $\mu \to \lambda_c C^{(1)}$ with $C^{(1)} = -5.516$. In the limit $\lambda_c \to 0$ the solution for V_0^* is given by

$$V_0^* = L_0(\rho), \quad \rho = \frac{Y^2}{2} + \cos x, \quad L(\rho) = L_0 + \int_1^{\rho} L_0'(\rho)\, \mathrm{d}\rho, \tag{29}$$

with

$$L(\rho) = \pm \frac{2\pi}{\int_0^{2\pi} \sqrt{2}(\rho - \cos x)^{\frac{1}{2}}\, \mathrm{d}x} \quad \text{for} \quad \begin{cases} Y > \sqrt{2}(\rho - \cos x)^{\frac{1}{2}} \\ Y < -\sqrt{2}(\rho - \cos x)^{\frac{1}{2}} \end{cases},$$

$$L(\rho) = L_0 \quad \text{inside the cat's eyes} \quad |Y| < \sqrt{2}(\rho - \cos x)^{\frac{1}{2}}.$$

In particular, from (28)

$$(V_0^*)_{-\infty}^{\infty} = \frac{1}{K_0}\left(v_M^+(\bar{\zeta}) - v_M^-(\bar{\zeta})\right) = \frac{1}{2}C^{(1)} \quad \text{for} \quad A \gg 1. \tag{30}$$

Thus, the problem for V_0^* shows that, even though the jump in the effective mean flow is zero, the separate components $\bar{u}_M, \bar{v}_M$ do suffer jumps across $\bar{\zeta}$.

For O(1) values of λ_c, a numerical solution of (27) is necessary, which can be found in, for example, [7].

The second-order problem gives rise to the set of equations,

$$\frac{\partial}{\partial x}(\tilde{u}_{01} + \tilde{u}_{10}) + \frac{\partial \tilde{w}_1}{\partial \eta} = 0,$$

$$N_1(\tilde{u}_1, \tilde{u}_0) + r\left(a_{0u}^2 - \left(1 + a_{0v}^2\right)\right) = -\alpha_0 \frac{\partial \tilde{p}_1}{\partial x} - \alpha_1 \frac{\partial \tilde{p}_0}{\partial x},$$

$$N_1(\tilde{v}_1, \tilde{v}_0) + 2ra_{0u}(1 + a_{0v}) = -\frac{m_0}{r}\frac{\partial \tilde{p}_1}{\partial x} - \frac{m_1}{r}\frac{\partial \tilde{p}_0}{\partial x},$$

with the operator N_1 defined by

$$N_1(\tilde{u}_1, \tilde{u}_0) = (\bar{B}_{10} + B_{01}\eta)\frac{\partial \tilde{u}_1}{\partial x} + (B_{20} + u_{10} + u_{01})\frac{\partial \tilde{u}_0}{\partial x} + \tilde{w}_0 \frac{\partial \tilde{u}_1}{\partial \eta} + (\tilde{w}_1 + a_{0w})\frac{\partial \tilde{u}_0}{\partial \eta} - \frac{\partial^2 \tilde{u}_1}{\partial \eta^2}.$$

These equations can be solved by setting

$$\tilde{u}_{01} + \tilde{u}_{10} = \frac{\partial \psi_1}{\partial \eta}, \quad \tilde{w}_1 = -\frac{\partial \psi_1}{\partial x}$$

to get

$$\tilde{u}_{01} + \tilde{u}_{10} = -B_{01} d(x, r), \tag{31}$$

where $d(x, r)$ can be determined by matching, although we do not need it explicitly.

The continuation below the critical layer requires the solution of the third-order problem, where $(\tilde{u}_2, \tilde{v}_2, \tilde{w}_2, \tilde{p}_2)$ satisfy the equations

$$\frac{\partial}{\partial x}(\tilde{u}_{02} + \tilde{u}_{11} + \tilde{u}_{20}) + \frac{\partial \tilde{w}_2}{\partial \eta} = 0, \tag{32a}$$

$$N_2(\tilde{u}_2, \tilde{u}_1, \tilde{u}_0) + \tilde{u}_0 a_{0u} - 2a_{ov}\tilde{v}_0 - 2\tilde{v}_0 + ra_{0u}\frac{\partial \tilde{u}_0}{\partial r} = -\left(\alpha_0 \frac{\partial \tilde{p}_2}{\partial x} + \alpha_1 \frac{\partial \tilde{p}_1}{\partial x} + \alpha_2 \frac{\partial \tilde{p}_0}{\partial x}\right), \tag{32b}$$

$$N_2(\tilde{v}_2, \tilde{v}_1, \tilde{v}_0) + a_{0u}\tilde{v}_0 + 2\tilde{u}_0 + a_{0u}\tilde{v}_0 + a_{0v}\tilde{u}_0 + ra_{0u}\frac{\partial \tilde{v}_0}{\partial r} = -\left(\frac{m_0}{r}\frac{\partial \tilde{p}_2}{\partial x} + \frac{m_1}{r}\frac{\partial \tilde{p}_1}{\partial x} + \frac{m_2}{r}\frac{\partial \tilde{p}_0}{\partial x}\right), \tag{32c}$$

$$(B_{01}\eta + \bar{B}_{10})\frac{\partial \tilde{w}_0}{\partial x} = -\frac{\partial \tilde{p}_2}{\partial \eta}. \tag{32d}$$

The operator N_2 above is defined by

$$N_2(\tilde{u}_2, \tilde{u}_1, \tilde{u}_0) = (B_{01}\eta + \bar{B}_{10})\frac{\partial \tilde{u}_2}{\partial x} + (B_{20} + \tilde{u}_{10} + \tilde{u}_{01})\frac{\partial \tilde{u}_1}{\partial x} + (B_{30} + \tilde{u}_{20} + \tilde{u}_{11} + \tilde{u}_{02})\frac{\partial \tilde{u}_0}{\partial x} + \tilde{w}_0\frac{\partial \tilde{u}_0}{\partial \eta}$$
$$+(w_1 + a_{0w})\frac{\partial \tilde{u}_1}{\partial \eta} + (\tilde{w}_2 + a_{1w}\eta)\frac{\partial \tilde{u}_0}{\partial \eta} - \frac{\partial^2 \tilde{u}_2}{\partial \eta^2} - \gamma_0^2\frac{\partial^2 \tilde{u}_0}{\partial x^2}.$$

If we let

$$\frac{\partial \psi_2}{\partial \eta} = \tilde{u}_{02} + \tilde{u}_{11} + \tilde{u}_{20}, \quad \frac{\partial \psi_2}{\partial x} = -\tilde{w}_2,$$

then, using (24, 31) and after some manipulation, it can be shown that ψ_2 satisfies

$$(B_{01}\eta + \bar{B}_{10})\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi_2}{\partial \eta^2}\right) + \tilde{w}_0\frac{\partial^3 \psi_2}{\partial \eta^3} - \frac{\partial^4 \psi_2}{\partial \eta^4} = -B_{03} - \gamma_0^2(B_{01}\eta + \bar{B}_{10})\tilde{w}_0$$
$$+\frac{2(1 + a_{0v})\gamma_0^2}{\alpha_0}\left[\frac{\partial \tilde{v}_0}{\partial \eta} - ra_{1v}\right]. \tag{33a}$$

The boundary conditions on ψ_2 to enable a match with the solutions outside the critical layer are

$$\frac{\partial \psi_2}{\partial \eta} \sim B_{03}\frac{\eta^3}{6} + \left(B_{12} + C_{02}^{\pm}\right)\frac{\eta^2}{2} + \left(D_{01}^{\pm} + C_{11}^{\pm} + B_{21}\right)\eta$$
$$+\frac{B_{01}\left(B_{12} + C_{02}^{\pm}\right) - B_{03}\bar{B}_{10}}{B_{01}^2}\left(\frac{\gamma_0^2 P_{00} A}{B_{01}}\right)\cos x \log|\eta| + U^{\pm}(x), \tag{33b}$$

as $\eta \to \pm\infty$. Here

$$C_{jk}^{\pm} = \left(\alpha_j \bar{u}_M^{(k)}(\bar{\zeta}) + \frac{m_j}{r}\bar{v}_M^{(k)}(\bar{\zeta})\right)^{\pm} \tag{34a}$$

$$D_{01}^{\pm} = \left(\alpha_0 \bar{u}_{m1}^{(k)}(\bar{\zeta}) + \frac{m_0}{r}\bar{v}_{m1}^{(k)}(\bar{\zeta})\right)^{\pm}. \tag{34b}$$

Equation 33a is the nonlinear counterpart of (17). We note that the $\partial/\partial r$ terms have cancelled out when working in terms of the cross-flow components, but they do not disappear in the equation for $\tilde{u}_2, \tilde{v}_2$. Also the terms in (33a) stemming from the Coriolis forces, cannot be removed, unlike in the linear case, primarily because the equation here is nonlinear. This and the $-B_{03}$ term (arising from the basic flow) provide the forcing in the resultant Haberman equation below. The $C_{02}^{\pm}, D_{01}^{\pm}$ contributions stem from the mean-flow terms in the inviscid region, as can be seen from (19).

The normalisations (26) together with

$$\frac{\partial^2\psi_2}{\partial\eta^2} = \frac{\partial^2\bar{\psi}_2}{\partial\eta^2} + \frac{\gamma_0^2}{B_{01}}\tilde{p}_0, \quad \bar{\psi}_2 = -C_*^3\left(B_{12} - \frac{B_{03}\bar{B}_{10}}{B_{01}}\right)\psi_2^*, \tag{35}$$

reduces (33) to the canonical form,

$$Y\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_2^*}{\partial Y^2}\right) + \sin x\frac{\partial^3\psi_2^*}{\partial Y^3} - \lambda_c\frac{\partial^4\psi_2^*}{\partial Y^4} = D_3\lambda_c + \sigma_*\left(1 - \frac{\partial V_0^*}{\partial Y}\right), \tag{36a}$$

$$\begin{aligned}\frac{\partial\psi_2^*}{\partial Y} &\sim -D_3\frac{Y^3}{6} + (1 + b_2^{\pm})\frac{Y^2}{2} + \left(d_1 + b_1^{\pm}\right)Y \\ &\quad + \left(1 + b_2^{\pm}\right)\cos x\log|Y| + YF_2(x) + F_1^{\pm}(x) \quad \text{as} \quad Y \to \pm\infty.\end{aligned} \tag{36b}$$

Here

$$\begin{aligned}&\alpha_c = \frac{B_{01}B_{12} - B_{03}\bar{B}_{10}}{B_{01}}, \quad \sigma_* = \frac{2(1 + a_{0v})\gamma_0^2K_0}{\alpha_0|B_{01}|\alpha_cC_*^3}, \ D_3 = \frac{B_{03}C_*}{\alpha_c}, \quad b_2^{\pm} = \frac{C_{02}^{\pm}}{\alpha_c}, \\ &-C_*D_1\alpha_c = B_{21} + \frac{B_{03}\bar{B}_{10}^2}{2B_{01}^2} - \frac{\bar{B}_{10}B_{12}}{B_{01}}, \quad -C_*\alpha_cb_1^{\pm} = -\frac{\bar{B}_{10}C_{02}^{\pm}}{B_{01}} + D_{01}^{\pm} + C_{11}^{\pm}.\end{aligned} \tag{37}$$

Equation 36 is a forced Haberman [7] equation and is different from previous studies on critical layers because of the forcing terms in (36a), and the extra $Y^3, b_2^{\pm}$ contributions in the boundary conditions. Nevertheless (36) does have properties similar to the Haberman [7] equation.

Identities relating the jump in the velocity to that in the mean vorticity can be derived as for the Haberman equation. These are obtained by integrating (36) with respect to Y, then over a period; see ([7]). It is found that

$$-\lambda_c\left(b_2^+ - b_2^-\right) = \frac{\sigma_*}{K_0}\left(\tilde{v}_M^+ - \tilde{v}_M^-\right), \tag{38}$$

and

$$\phi - 2\lambda_c(b_1^+ - b_1^-) = -\frac{\sigma_*}{\pi}\int_0^{2\pi}(B_0^+ - B_0^-)\,\mathrm{d}x. \tag{39}$$

The quantity $\phi = \frac{1}{\pi}\int_0^{2\pi}\sin x(F_1^+ - F_1^-)\,\mathrm{d}x$ is the jump in the $\sin x$ component of $\partial\psi_2/\partial Y$

We consider next how the jump in the velocity modifies the eigenrelations.

4.3 Nonlinear eigenrelations

The leading-order wall displacement is basically the same as in the linear case. The expansions in the wall layer ($z = \epsilon^4\xi$) are

$$\begin{aligned}u &= \epsilon r\bar{u}(0) + \epsilon^2u_0 + \cdots, \\ v &= \epsilon r\bar{v}(0) + \epsilon^2v_0 + \cdots, \\ w &= \epsilon^3w_0 + \cdots, \\ p &= \epsilon^3p_0 + \cdots.\end{aligned}$$

Solutions for the fundamental are as in the linear case, see (11), and after matching with the $\bar{w}_1$ component in the inviscid regions gives

$$(\bar{w}_1)(\zeta = 0) = \frac{-3\mathrm{Ai}'(0)}{\lambda^{\frac{1}{3}}} A \cos\left(x + \frac{\pi}{3}\right). \tag{40}$$

The critical-layer problem (36) determines the jump conditions on $\partial\bar{w}_1/\partial\eta$ and with the above normalisations we find that

$$\left(w'_{1c}\right)^+_- = C_*^2(-\phi)\frac{B_{12}B_{01} - B_{03}\bar{B}_{10}}{B_{01}}, \tag{41a}$$

$$(w'_{1s})^+_- = 0. \tag{41b}$$

Hence the resulting eigenrelations are

$$-\frac{3\mathrm{Ai}'(0)}{2\lambda^{\frac{1}{3}}} = (-\phi) w_0^2(\bar{\zeta})\left(\frac{B_{12}B_{01} - B_{03}\bar{B}_{10}}{B_{01}^2}\right), \tag{42a}$$

and

$$\frac{3\sqrt{3}}{2\lambda^{\frac{1}{3}}}\mathrm{Ai}'(0) = 2\left(\alpha_0\alpha_1 + \frac{m_0 m_1}{r^2}\right)\int_0^\infty w_0^2 \,\mathrm{d}\zeta - \int_0^\infty \left[U''_{0B}(U_{0M} + U_{1B}) - U_{0B}\left(U''_{0M} + U''_{1B}\right)\right]\frac{w_0^2(\zeta)}{U_{0B}^2}\,\mathrm{d}\zeta. \tag{42b}$$

Now

$$\frac{\alpha_c}{B_{01}} = \frac{B_{12}B_{01} - B_{03}\bar{B}_{10}}{B_{01}^2} = \frac{r}{m_0}\left(\alpha_1 - \frac{m_1\alpha_0}{m_0}\right)\left(\frac{a_{2u}B_{01} - B_{03}a_{0u}}{B_{01}^2}\right)m_0 - \frac{B_{03}C_{00}}{B_{01}^2}. \tag{43}$$

The last term in (43) involving C_{00} is not present in the linear case and represents the effects of the mean-flow term.

From (16), (40), (42) we finally obtain

$$\left(\frac{\alpha_1}{m_0} - \frac{\alpha_0 m_1}{m_0^2}\right) = 17.5 r^{-\frac{1}{3}}\left(\frac{\pi}{-\phi}\right) + \frac{B_{03}C_{00}}{m_0(a_{2u}B_{01} - B_{03}a_{0u})} \tag{44a}$$

$$\left(\alpha_0\alpha_1 + \frac{m_0 m_1}{r^2}\right) = -9.2 r^{-\frac{1}{3}}\left(\frac{\pi}{-\phi}\right) - \frac{1}{2J_1}\left[\frac{J_2}{m_0}\left(\frac{B_{03}C_{00}}{a_{2u}B_{01} - B_{03}a_{0u}}\right) - J_3\right], \tag{44b}$$

where

$$J_1 = \int_0^\infty w_0^2\,\mathrm{d}\zeta, \quad J_2 = m_0 \fint_0^\infty \frac{U_{0B}\bar{u}'' - U''_{0B}\bar{u}}{U_{0B}^2}\,\mathrm{d}\zeta, \quad J_3 = \fint_0^\infty \frac{U''_{0B}u_{0M} - U_{0B}u''_{0M}}{U_{0B}^2}\,\mathrm{d}\zeta.$$

The corrections to the wave number and wave angle given by (44) are the main results in this paper and show the dependence of the wavenumber and angle on the neutral amplitude A via the phase jump ϕ, and the mean flow terms. For a given A and hence λ_c, a numerical solution of the forced Haberman equation (36a) is required to determine ϕ. If $\lambda_c \to \infty$, corresponding to $A \to 0$, it can be shown that $\phi \to -\pi$ and a match with linear theory is achieved. For large amplitudes $A \gg 1$ then $\lambda_c \to 0$ and the solution of the

Haberman equation gives rise to the cat's eye structure with ϕ approaching zero. The limiting structure is akin to that of the strongly nonlinear critical layer [8]. The full details of this limit are omitted here, (these are available by request from the author), but the analysis suggests that there is a cross-over from the inflexional instability to Benney–Bergeron type modes. These are amplitude-dependent stationary neutral modes.

5 Discussion

In this paper we have discussed critical layers in rotating-disk flow and have shown how the calculation of the wavenumber and waveangle depend crucially on the properties of the nonlinear critical layer. For very small amplitudes $A(\ll R^{-\frac{1}{3}})$ the stationary modes correspond to inflexional modes and are the same as those first discussed in [1]. For increased amplitudes $A \sim R^{-\frac{1}{3}}$ these modes are again inflexional but are now amplitude-dependent and require the solution of the forced Haberman equation with novel boundary conditions. For larger amplitudes $A \gg R^{-\frac{1}{3}}$, specifically $A \sim R^{-\frac{2}{9}}$, there is a cross-over from the inflexional modes to Benney–Bergeron-type modes. If we assume $C_{00} = 0$ in (44a) and $b_2^+ = b_2^-$ to leading order in (38), then in this second stage the wavenumber is decreasing as the amplitude increases, and an O(1) change in the wavenumber is predicted when $A \sim R^{-\frac{2}{9}}$. The decreased wavenumber, corresponding to fewer vortices, is in qualitative agreement with experiments.

In GSW it was suggested that viscosity would account for the discrepancy between the observed and calculated values of the wavenumber from linear theory. The influence of nonlinearity, as shown here, can significantly alter the wavenumber and it is possible that the experimentally observed modes are highly nonlinear stationary modes of the type discussed here. Of course a more quantitative comparison with experiments requires a solution of the nonlinear forced Haberman problem which is left open.

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